ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 297 (2004) 696–719

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Inheritance of surjectivity for partial differential operators on spaces of real analytic functions

Michael Langenbruch

Faculty V, Institute of Mathematics, University of Oldenburg, D-26111 Oldenburg, Germany

Received 30 January 2004

Submitted by R.M. Aron

Dedicated to Professor John Horváth on the occasion of his 80th birthday

Abstract

For an open set $\Omega \subset \mathbb{R}^n$ let $A(\Omega)$ be the space of real analytic functions on Ω. Improving our previous results, we prove a new quantitative characterization of the linear partial differential operators $P(D)$ which are surjective on $A(\Omega)$. This implies that $P(D)$ is surjective on $A(\mathbb{R}^n)$ if $P(D)$ is surjective on $A(\Omega)$ for some $\Omega \neq \emptyset$. Further inheritance properties for the surjectivity of $P(D)$ on $A(\Omega)$ are also obtained.

Keywords: Partial differential operators; Elementary solutions; Singular support; Surjectivity; Real analytic functions

1. Introduction

In this paper, we continue the study of the basic question when

$$P(D): A(\Omega) \to A(\Omega) \text{ is surjective.} \tag{1.1}$$

Here $P(D)$ is a partial differential operator with constant coefficients, $\Omega \subset \mathbb{R}^n$ is an open set and $A(\Omega)$ is the space of real analytic functions on Ω.

Solutions to this problem have been given mainly by two methods: Hörmander [8] has characterized (1.1) for *convex* open sets Ω by means of a Phragmen–Lindelöf condition

E-mail address: michael.langenbruch@uni-oldenburg.de.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.04.035

valid on the complex variety of P_m. His method has been adapted by several authors for further studies on this problem (see Miwa [20], Andreotti and Nacinovich [2], Zampieri [25], Braun [4] and the recent papers of Braun et al. [5,6]).

Hörmander's criterion is restricted to convex sets Ω by the use of Fourier theory. On the other hand, Kawai [13] used so-called "good elementary solutions" for $P(D)$ to prove (1.1) for locally hyperbolic operators on special, not necessarily convex bounded open sets Ω (see Kaneko [10–12] for the case of unbounded open sets and further results in the spirit of Kawai's work, and Andersson [1] for $\Omega = \mathbb{R}^n$).

In Langenbruch [16] we recently clarified the role of fundamental solutions for our problem, and we gave several characterizations of (1.1) by means of elementary solutions and also by conditions of type (A_Ω) (see Theorem 2.3 below).

In the present paper, a quantitative version of the latter characterization will be proved. Using this new condition and a result of Hörmander [8], we will show that

$$\begin{aligned} &P(D) \text{ is surjective on } A(\mathbb{R}^n) \\ &\quad \text{if } P(D) \text{ is surjective on } A(\Omega) \text{ for some } \Omega \neq \emptyset. \end{aligned} \tag{1.2}$$

For convex Ω, this is one of the main results of Hörmander [8], while the question had been open for general Ω. Thus, surjectivity of $P(D)$ on $A(\mathbb{R}^n)$ is a general necessary condition for (1.1).

We also show that surjectivity of partial differential operators on real analytic functions is inherited similarly as for operators on C^∞-functions. In fact, if $P(D)$ is surjective on $A(\Omega_j)$ for any $j \in J$ then $P(D)$ is surjective on $A(\Omega)$ for

$$\Omega := \Bigl(\bigcap_{j \in J} \Omega_j\Bigr)^\circ$$

and

$$\begin{aligned} \Omega &:= \liminf_j{}^\circ \Omega_j \\ &:= \bigl\{\xi \in \mathbb{R}^n \mid \exists \varepsilon > 0\colon\ B_\varepsilon(\xi) \subset \Omega_j \text{ for any but finitely many } j\bigr\}. \end{aligned}$$

Also, if $P(D)$ is surjective on $A(\Omega)$, then $P(D)$ is surjective on $A(\Omega_\varepsilon)$ for any $\varepsilon > 0$, where

$$\Omega_\varepsilon := \bigl\{\xi \in \Omega \mid \operatorname{dist}(\xi, \partial\Omega) > \varepsilon\bigr\}.$$

Next, we obtain the following result of Andreotti and Nacinovich [2] and Zampieri [25]: For convex Ω, (1.1) holds if and only if $P(D)$ is surjective on $A(H)$ for any tangent halfspace H of Ω.

We finally show an extension of this result for homogeneous operators $P(D)$ and open sets Ω with C^1-boundary: In this case, $P(D)$ is surjective on $A(\operatorname{conv}(\Omega))$ and on $A(H)$ for any tangent halfspace of Ω if (1.1) holds.

The paper is organized as follows: The necessary notations and results from Langenbruch [16] are recalled in Section 2. Section 3 contains the main technical tools of this paper (see Proposition 3.2 and Theorem 3.3) and the proofs of the claims stated above with the exception of our main result (1.2), which is proved in Section 4.

2. Preliminaries

In this section, we will introduce some useful notation and we will recall the basic results from Langenbruch [16] which are needed in the following.

In this paper, $n \in \mathbb{N}$ always is at least 2. Ω and ω are open sets in $\mathbb{R}^n$. We also assume that Ω is connected.

The real analytic functions on Ω are denoted by $A(\Omega)$. $P(D)$ is always a partial differential operator in n variables with constant coefficients. Our proofs will be based on harmonic functions in $(n+1)$ variables. The corresponding notations and facts will be given now.

A point in $\mathbb{R}^{n+1}$ is written as $(x, y) \in \mathbb{R}^n \times \mathbb{R}$. $\Delta = \sum_{k \leqslant n} (\partial/\partial x_k)^2 + (\partial/\partial y)^2$ denotes the Laplace operator on $\mathbb{R}^{n+1}$. The harmonic germs near a set $S \subset \mathbb{R}^{n+1}$ are denoted by $C_\Delta(S)$. Of course, $P(D) = P(D_x)$ also operates on the harmonic germs, and in fact we will solve the equation $P(D_x) f = g$ for harmonic germs f and g rather than for real analytic functions f and g. For this, we need the following basic lemmata (compare Remarks 3.1 and 3.2 in Langenbruch [16]). For compact sets $Q \subset L \subset \mathbb{R}^{n+1}$ let

$$R_L^Q : C_\Delta(L) \to C_\Delta(Q)$$

be the canonical mapping defined by restriction.

Lemma 2.1. *Let $Q \subset L \subset \mathbb{R}^{n+1}$ be compact sets such that*

$$\mathbb{R}^{n+1} \setminus Q \text{ does not have a bounded component} \tag{2.1}$$

(*and the same for L*). *Then*

$$P(D) C_\Delta(Q) \supset R_L^Q \big(C_\Delta(L) \big)$$

if for any bounded set B in $C_\Delta(Q)'_b$ the set

$$\tilde{B} := \big\{ \mu \in C_\Delta(Q)' \mid P(-D)\mu \in B \big\}$$

is bounded in $C_\Delta(L)'_b$.

Proof. The condition implies that the mapping

$${}^t R_L^Q \circ P(-D)^{-1} : P(-D) C_\Delta(Q)' \subset C_\Delta(Q)'_b \to C_\Delta(L)'_b$$

is defined and sequentially continuous (and hence continuous) on the metric space $P(-D) C_\Delta(Q)' \subset C_\Delta(Q)'_b$. The proof is completed by a standard Hahn–Banach argument. □

Lemma 2.2. *Let $Q \subset \mathbb{R}^{n+1}$ be compact with* (2.1). *Then for any bounded set B in $C_\Delta(Q)'_b$ the set*

$$\tilde{B} := \big\{ \mu \in C_\Delta(Q)' \mid P(-D)\mu \in B \big\}$$

is bounded in $C_\Delta(\operatorname{conv}(Q))'_b$.

To apply Lemma 2.1 we need an appropriate representation for $C_\Delta(Q)'_b$. This is provided by the Grothendieck–Tillmann duality: Let

$$G(x,y) := -\left|(x,y)\right|^{1-n} \big/ \big((n-1)c_{n+1}\big)$$

be the canonical even elementary solution of the Laplacian (see Hörmander [9], and recall that $(n+1) \geqslant 3$). For $Q \subset \mathbb{R}^{n+1}$ compact let

$$C_{\Delta,0}(\mathbb{R}^{n+1} \setminus Q) := \Big\{ f \in C_\Delta(\mathbb{R}^{n+1} \setminus Q) \;\Big|\; \lim_{\xi \to \infty} f(\xi) = 0 \Big\}$$

endowed with the topology of $C(\mathbb{R}^{n+1} \setminus Q)$. $C_{\Delta,0}(\mathbb{R}^{n+1} \setminus Q)$ is a Fréchet space.

$$C(V \setminus Q) \text{ also induces the topology of } C_{\Delta,0}(\mathbb{R}^{n+1} \setminus Q) \tag{2.2}$$

if V is an open neighborhood of Q. Let

$$\varkappa(\mu)(x,y) := u_\mu(x,y) := \big\langle \mu_{(s,t)}, G(s-x, t-y) \big\rangle \quad \text{for } \mu \in C_\Delta(Q)'_b.$$

Then we have the topological isomorphisms

$$\varkappa : C_\Delta(Q)'_b \to C_{\Delta,0}(\mathbb{R}^{n+1} \setminus Q) \cong C_\Delta(\mathbb{R}^{n+1} \setminus Q)/C_\Delta(\mathbb{R}^{n+1}) \tag{2.3}$$

by the Grothendieck–Tillmann duality (Grothendieck [7, Theorem 4], Mantovani and Spagnolo [17], Tillmann [22, Satz 6]).

As a first application of (2.3) we notice the following: Let $Q, L \subset \mathbb{R}^{n+1}$ be compact sets both satisfying (2.1). Then a set

$$\begin{aligned} &B \subset C_\Delta(Q \cap L)'_b \text{ is bounded} \\ &\quad \text{if } B \subset C_\Delta(Q)'_b \text{ and } B \subset C_\Delta(L)'_b \text{ is bounded.} \end{aligned} \tag{2.4}$$

To characterize the surjective partial differential operators on $A(\Omega)$ the useful technical conditions $(\underline{A_\Omega})$ and $(\overline{A_\Omega})$ were introduced in Langenbruch [16]. We summarize the definitions and the characterization in the following theorem (see Theorem 4.6 in Langenbruch [16]). For $T \in \mathbb{R}$ let

$$U\big(\Omega \times \{T\}\big) := \big\{ V \subset \mathbb{R}^{n+1} \text{ open } \mid V \cap \big(\mathbb{R}^n \times \{T\}\big) = \Omega \times \{T\} \big\}$$

and $U(\Omega) := U(\Omega \times \{0\})$.

Theorem 2.3. *The following statements are equivalent*:

(i) *$P(D)$ is surjective on $A(\Omega)$.*

(ii) *$P(D)$ satisfies the following condition $(\underline{A_\Omega})$: for any $\omega \Subset \Omega$ there is $\tilde\omega \Subset \Omega$ with $\omega \Subset \tilde\omega$ such that for any $\hat\omega \Subset \Omega$ and any $\xi \in \partial\tilde\omega$ there is $\delta > 0$ such that for any $0 < T \leqslant \delta$ there are $V \in U(\hat\omega \times \{T\})$ and $E \in C_\Delta(W)$, $W := V \cup (\omega \times]T-\delta, T+\delta[)$, such that*

$$P(D)E = G(\cdot - \xi, \,)|_W.$$

(iii) *$P(D)$ satisfies the following condition $(\overline{A_\Omega})$: for any $\omega \Subset \Omega$ there are $\tilde\omega \Subset \Omega$ with $\omega \Subset \tilde\omega$ and $1 \geqslant \delta_0 > 0$ such that for any $\xi \in \Omega \setminus \tilde\omega$ and any $0 < T \leqslant 1$ there are $V \in U(\Omega \times \{T\})$ and $E \in C_\Delta(W)$, $W := V \cup (\omega \times]T-\delta_0, T+\delta_0[)$, such that*

$$P(D)E = G(\cdot - \xi, \,)|_W.$$

3. The basic results

We will prove the main technical results in this section (see Proposition 3.2 and Theorem 3.3), leading to the new condition (A_Ω) (see Theorem 3.4) which is equivalent to the surjectivity of $P(D)$ on $A(\Omega)$ and can be considered as a *quantitative* intermediate version of the conditions $(\underline{A_\Omega})$ and $(\overline{A_\Omega})$ from Theorem 2.3. We will then prove some consequences concerning the inheritance of surjectivity of partial differential operators on spaces of real analytic functions including the results stated already in the introduction.

The results of this section are intimately connected to the possibility of shifting subsets of Ω within Ω. We now introduce the corresponding notation: For a compact $K \subset \Omega$ let

$$S(K,\Omega) := \{\xi \in \mathbb{R}^n \mid \xi + K \subset \Omega\}$$

and let $S_0(K,\Omega)$ be the component of 0 in $S(K,\Omega)$. $S_0(K,\Omega)$ is open and pathconnected since $S(K,\Omega)$ is open. Let

$$\Omega_\varepsilon := \bigl\{\xi \in \Omega \mid \operatorname{dist}(\xi, \partial\Omega) > \varepsilon\bigr\}.$$

The Ω-hull K_Ω of a compact $K \subset \Omega$ is defined by

$$K_\Omega := \bigl\{x \in \mathbb{R}^n \mid x + S_0(K,\Omega) \subset \Omega\bigr\} = \bigcap_{\xi \in S_0(K,\Omega)} (\Omega - \xi).$$

K is contained in K_Ω by the definition of $S_0(K,\Omega)$, and K_Ω is contained in Ω since $0 \in S_0(K,\Omega)$. Let

$$K_\Omega^\varepsilon := \bigcap_{\xi \in S_0(K,\Omega_\varepsilon)} (\Omega - \xi)$$

and let

$$B_C(x) := \bigl\{\xi \in \mathbb{R}^n \mid \|x - \xi\| < C\bigr\}.$$

Lemma 3.1. *Let $K \subset \Omega$ be compact.*

(a) *$K_\Omega \subset \Omega_\varepsilon$ if $K \subset \Omega_\varepsilon$.*
(b) *For any $C \geqslant 1$ and any $\varepsilon > 0$ there is $\eta > 0$ such that*

$$B_C(0) \cap \bigl(K + \overline{B_\eta(0)}\bigr)_\Omega^\eta \subset K_\Omega + B_\varepsilon(0).$$

Proof. (a) Let $K \subset \Omega_\varepsilon$. Then $B_{\tilde\varepsilon}(0) + K \subset \Omega$ for some $\tilde\varepsilon > \varepsilon$ and therefore $B_{\tilde\varepsilon}(0)$ is contained in $S_0(K,\Omega)$. By the definition of K_Ω we get $K_\Omega + B_{\tilde\varepsilon}(0) \subset \Omega$, that is, $K_\Omega \subset \Omega_\varepsilon$.

(b) If the claim were not true, there are $C \geqslant 1$ and $\varepsilon > 0$ and a sequence

$$x_n \in B_C(0) \cap \bigl(K + \overline{B_{1/n}(0)}\bigr)_\Omega^{1/n} \tag{3.1}$$

such that

$$x_n \notin K_\Omega + B_\varepsilon(0). \tag{3.2}$$

Since the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, we may suppose that $x := \lim_{n\to\infty} x_n$ exists. We will show that $x \in K_\Omega$ contradicting (3.2). So let $\xi \in S_0(K,\Omega)$. Then there is a path

$$\gamma : [0,1] \to S(K,\Omega)$$

connecting 0 and ξ. Moreover, there is n_0 such that

$$\overline{B_{2/n_0}\big(\gamma(t)\big)} + K \subset \Omega_{1/n_0} \quad \text{for any } t \in [0,1].$$

This implies that

$$B_{1/n_0}(\xi) \subset S_0\big(K + \overline{B_{1/n}(0)}, \Omega_{1/n}\big) \quad \text{for any } n \geqslant n_0.$$

By (3.1) we get $B_{1/n_0}(\xi) + x_n \subset \Omega$ and therefore $x \in K_\Omega$ since

$$x + \xi = x_n + \big[(x - x_n) + \xi\big] \subset \Omega. \qquad \square$$

The following decomposition is the main technical tool of this paper. It is formulated in such a way that it can be applied repeatedly. Let

$$I(c) :=]-c, c[\quad \text{and} \quad J(c) := [-c, c] \quad \text{for } c > 0.$$

Proposition 3.2. *Let $X, K, \tilde{K}, Z, \tilde{Z}$ (and $\omega, \tilde{\omega}, \omega_1, \tilde{\omega}_1$, respectively) be (relatively) compact subsets of Ω and let $\delta > \gamma > 0$ such that*

$$X \subset K \subset \omega \subset \tilde{\omega} \Subset \tilde{K}^\circ, \qquad X \subset Z \subset \omega_1 \subset \tilde{\omega}_1 \Subset \tilde{Z}^\circ \quad \textit{and} \quad 0 < \delta < \delta_0/4,$$

where $\tilde{\omega}$ and $\delta_0 > 0$ are chosen for ω by $(\overline{A_\Omega})$. For $Q := (X \times J(\delta)) \cup (Z \times J(\beta))$ let B be bounded in $C_\Delta(Q)'_b$ and let

$$\tilde{B} := \big\{\mu \in C_\Delta(Q)' \mid P(-D)\mu \in B\big\}.$$

Then for sufficiently small $0 < \beta < \gamma$, $\tilde{B}$ can be decomposed as

$$\tilde{B} \subset \tilde{B}_I + \tilde{B}_{II},$$

where

$$\tilde{B}_I \subset C_\Delta\big(\tilde{Z} \times J(\gamma)\big)'_b \quad \textit{and} \quad \tilde{B}_{II} \subset C_\Delta\big((\tilde{K} \cap \tilde{Z}) \times J(\delta)\big)'_b$$

are bounded, and also

$$P(-D)\tilde{B}_{II} \subset C_\Delta\Big(\big((\tilde{K} \cap \tilde{Z}) \times J(\gamma)\big) \cup \big(X \times J(\delta)\big)\Big)'_b$$

is bounded, if either

$$\tilde{B} \textit{ is bounded in } C_\Delta\big(Z \times J(\delta)\big)'_b, \tag{3.3}$$

or

$$\tilde{K} \subset Z \textit{ and } \tilde{\omega}_1 \textit{ is chosen for } \omega_1 \textit{ by } (\overline{A_\Omega}). \tag{3.4}$$

Proof. The proof is an improved version of parts of the proof of Theorem 4.4 in Langenbruch [16].

(a) By Lemma 2.2 and (2.3) we get

$$\{u_\mu \mid \mu \in \tilde{B}\} \text{ is bounded in } C_\Delta\big(\mathbb{R}^{n+1} \setminus \big(\operatorname{conv}(Z) \times J(\delta)\big)\big). \tag{3.5}$$

Using $(\overline{A_\Omega})$ we now get additional bounds if β is sufficiently small: There is a neighborhood V of $\partial(\tilde{\omega} \cap \tilde{\omega}_1) \times ([-\delta, -\gamma/2] \cup [\gamma/2, \delta])$ such that for any $a \in \mathbb{N}_0^{n+1}$,

$$\sup\big\{\big|D^a u_\mu(x, y)\big| \mid (x, y) \in V,\ \mu \in \tilde{B}\big\} \leqslant C_a < \infty. \tag{3.6}$$

For $\xi \in \partial\tilde{\omega}$ and $0 < T \leqslant 2\delta$ we choose $V_{\xi,T} \in U(\Omega \times \{T\})$ and $E_{\xi,T} \in C_\Delta(V_{\xi,T} \cup (\omega \times]T - \delta_0, T + \delta_0[))$ by $(\overline{A_\Omega})$. Since $K \subset \omega$ and $Z \subset \Omega$ are compact there are $\rho > 0$ and $0 < \nu_{\xi,T} < \delta/4$ such that

$$K + \overline{B_\varrho(0)} \subset \omega \quad \text{and} \quad \big(Z + \overline{B_\varrho(0)}\big) \times [T - 2\nu_{\xi,T}, T + 2\nu_{\xi,T}] \subset V_{\xi,T}. \tag{3.7}$$

For $(x, y) \in B_\varrho(0) \times I(\nu_{\xi,T})$ and $a \in \mathbb{N}_0^{n+1}$ we thus get for $\mu \in \tilde{B}$,

$$\begin{aligned} \big|D^a u_\mu(x + \xi, y - T)\big| &= \big|\big\langle \mu_{(s,t)}, D^a G(s - \xi - x, t + T - y)\big\rangle\big| &(3.8)\\ &= \big|\big\langle \mu_{(s,t)}, P(D_s) D^a E_{\xi,T}(s - x, t + T - y)\big\rangle\big| \\ &= \big|\big\langle P(-D)\mu_{(s,t)}, D^a E_{\xi,T}(s - x, t + T - y)\big\rangle\big| &(3.9)\end{aligned}$$

if $0 < \beta \leqslant \nu_{\xi,T}$ since $\delta \leqslant \delta_0/4$. This implies that

$$\sup\big\{\big|u_\mu(x, y)\big| \mid (x, y) \in B_\rho(\xi) \times]-T - \nu_{\xi,T}, -T + \nu_{\xi,T}[,\ \mu \in \tilde{B}\big\} \leqslant C,$$

since $P(-D)\mu \in B$, where B is bounded in $C_\Delta(Q)'_b$, and since

$$\big\{D^a E_{\xi,T}(\cdot - x, \cdot + T - y) \mid x \in B_\rho(0),\ y \in I(\nu_{\xi,T})\big\}$$

is bounded in $C_\Delta(Q)$ by (3.7) since $0 < \beta \leqslant \nu_{\xi,T}$.

Applying (3.9) for $u_\mu(\cdot, -\cdot)$ instead of u_μ we get a similar bound for u_μ on $B_\rho(\xi) \times]T - \nu_{\xi,T}, T + \nu_{\xi,T}[$.

In case of (3.4) we have $\tilde{\omega} \cap \tilde{\omega}_1 = \tilde{\omega}$, and the claim then follows since $\partial\tilde{\omega} \times ([-\delta, -\gamma/2] \cup [\gamma/2, \delta])$ is compact.

On the other hand, using only (3.8), (3.3) implies a bound like (3.6) near $\partial\tilde{\omega}_1 \times (\pm[\gamma/2, \delta])$ since $Z \subset \omega_1$. Thus, (3.6) also holds in this case.

(b) For $\mu \in \tilde{B}$ we now decompose u_μ into $v_\mu + w_\mu$, keeping track of uniform estimates. Choose a neighborhood $U_1 \Subset (\tilde{Z} \cap \tilde{K})^\circ$ of $\partial(\tilde{\omega} \cap \tilde{\omega}_1)$ such that $U_1 \times (\pm[\gamma/2, \delta]) \subset V$ for V from (3.6). We set $U := U_1 \cup (\tilde{\omega} \cap \tilde{\omega}_1)$ and choose $\varphi \in C_0^\infty(U \times I(2\delta))$ such that $\varphi = 1$ near $\overline{(\tilde{\omega} \cap \tilde{\omega}_1)} \times J(\delta)$. Let $0 < \beta < \gamma/2$ and set $\tilde{f}_\mu := \Delta(\varphi u_\mu)|_W$ for $W := \mathbb{R}^n \times (\mathbb{R} \setminus J(\gamma/2))$. By (3.5) and (3.6) $\tilde{f}_\mu$ can be extended trivially (i.e., by the value 0) to a bounded function f_μ on $\mathbb{R}^{n+1}$ such that

$$\operatorname{supp} f_\mu \subset W_1 := V \cup \big((\tilde{\omega} \cap \tilde{\omega}_1) \times \big(I(2\delta) \setminus J(\delta)\big)\big).$$

Since $D^a u_\mu$ is locally uniformly bounded on W_1 by (3.5) and (3.6) and since $P(-D_x) f_\mu$ is the trivial extension of $P(-D_x)\tilde{f}_\mu$, $P(-D_x) f_\mu$ is also bounded and

$$\sup\big\{\|f_\mu\|_\infty + \big\|P(-D) f_\mu\big\|_\infty \mid \mu \in \tilde{B}\big\} \leqslant C_1. \tag{3.10}$$

Let $g_\mu := G * f_\mu$. Then

$$\sup\big\{\|g_\mu\|_\infty + \big\|P(-D) g_\mu\big\|_\infty \mid \mu \in \tilde{B}\big\} \leqslant C_2 \tag{3.11}$$

by (3.10) since W_1 is bounded. Set

$$v_\mu := (\varphi u_\mu - g_\mu)|_{\mathbb{R}^{n+1} \setminus ((\tilde{K} \cap \tilde{Z}) \times J(\delta))}$$

and

$$w_\mu := \big((1 - \varphi) u_\mu + g_\mu\big)\big|_{\mathbb{R}^{n+1} \setminus (\tilde{Z} \times J(\gamma))},$$

where $(1-\varphi)u_\mu$ is defined by trivial extension. Then $v_\mu \in C_{\Delta,0}(\mathbb{R}^{n+1} \setminus ((\tilde{K} \cap \tilde{Z}) \times J(\delta)))$ and $w_\mu \in C_{\Delta,0}(\mathbb{R}^{n+1} \setminus (\tilde{Z} \times J(\gamma)))$. Moreover, $u_\mu = v_\mu + w_\mu$ outside $\tilde{Z} \times J(\delta)$.

(c) By (3.5), (3.6) and (3.11),

$$\{v_\mu \mid \mu \in \tilde{B}\} \text{ is bounded in } C_\Delta\big(\mathbb{R}^{n+1} \setminus \big((\tilde{K} \cap \tilde{Z}) \times J(\delta)\big)\big). \tag{3.12}$$

Moreover, since $P(-D)u_\mu$ is contained and bounded in $C_\Delta(\mathbb{R}^{n+1} \setminus Q)$, we also have $P(-D)v_\mu \in C_{\Delta,0}(\mathbb{R}^{n+1} \setminus S)$ for $S := ((\tilde{K} \cap \tilde{Z}) \times J(\gamma) \cup (X \times J(\delta))$ and

$$\big\{P(-D)v_\mu \mid \mu \in \tilde{B}\big\} \text{ is bounded in } C_{\Delta,0}(\mathbb{R}^{n+1} \setminus S) \tag{3.13}$$

by (3.6) and (3.11).

(d) By (3.6), (3.11) and the assumption on B (and $\tilde{B}$),

$$\big\{P(-D)w_\mu \mid \mu \in \tilde{B}\big\} \text{ is bounded in } C_\Delta\big(\mathbb{R}^{n+1} \setminus \big(Z \times J(\gamma)\big)\big). \tag{3.14}$$

Hence,

$$\{w_\mu \mid \mu \in \tilde{B}\} \text{ is bounded in } C_\Delta\big(\mathbb{R}^{n+1} \setminus \big(\operatorname{conv}(\tilde{Z}) \times J(\gamma)\big)\big) \tag{3.15}$$

by Lemma 2.2. By (2.3) and (2.4) this directly implies that

$$\{w_\mu \mid \mu \in \tilde{B}\} \text{ is bounded in } C_\Delta\big(\mathbb{R}^{n+1} \setminus \big(\tilde{Z} \times J(\gamma)\big)\big) \tag{3.16}$$

if (3.3) holds. If (3.4) is assumed instead, (3.16) follows as above from (3.14) and $(\overline{A_\Omega})$ (and (3.15)), if we assume (as we may) that $\gamma > 0$ is sufficiently small. In view of (3.12), (3.13), (3.16) and (2.3), the proposition is proved. $\square$

Since the decomposition problem in $C_\Delta(Q)'$ has been solved in Proposition 3.2 we can now use a suitable shifting procedure to prove the following result which is basic for this paper and which improves on Theorem 4.4(a) in Langenbruch [16]. Theorem 3.3(b)–(d) will be used in Section 4 for $X := \{z\}$, $z \in \Omega$, to show that $P(D)$ is surjective on $A(\mathbb{R}^n)$ if $P(D)$ is surjective on $A(\Omega)$ for some non-void Ω.

Theorem 3.3. *Let $P(D)$ be surjective on $A(\Omega)$.*

(a) *For any compact $X \subset \Omega$ there is $C > 0$ such that for any $\varepsilon > 0$ there are $\eta > 0$, $\delta_0 > 0$ and a compact $\tilde{X}_\varepsilon$ with*

$$X \subset \tilde{X}_\varepsilon \subset \big(X_\Omega + B_\varepsilon(0)\big) \cap B_C(0)$$

such that for any $0 < \delta \leqslant \delta_0$, any compact $Y \subset \Omega$ and any $0 < \gamma < \delta$ there are a compact $\tilde{Y} \subset \Omega$ with $Y \subset \tilde{Y}$ and $0 < \beta < \gamma$ such that

$$P(D)C_\Delta\big(\big(\big(X + \overline{B_\eta(0)}\big) \times J(\delta)\big) \cup \big(Y \times J(\beta)\big)\big)$$
$$\supset C_\Delta\big(\big(\tilde{X}_\varepsilon \times J(\delta)\big) \cup \big(\tilde{Y} \times J(\gamma)\big)\big)\big|_{((X+\overline{B_\eta(0)})\times J(\delta))\cup(Y\times J(\beta))}.$$

(b) *We may choose $\tilde{X}_\varepsilon$ in* (a) *such that*

$$\tilde{X}_\varepsilon \subset \bigcap_{V \cap X \neq \emptyset} V, \tag{3.17}$$

where V are the components of $(X_\Omega + B_\varepsilon(0)) \cap B_C(0)$.

(c) *There is* $C > 0$ *such that for any* $\xi \in S_0(X, \Omega)$ *we may choose* $(\xi + X)\tilde{}_\varepsilon$ *in* (a) *such that*

$$(\xi + X)\tilde{}_\varepsilon \subset \xi + \big[\big(X_\Omega + B_\varepsilon(0)\big) \cap B_C(0)\big].$$

(d) *We may take* $\tilde{Y} = Y$ *in* (a), *if* Y *is convex and* $\tilde{X}_\varepsilon \subset Y$.

Proof. (a)(i) It is sufficient to show that (for the quantifiers as in (a)) we may choose

$$\tilde{X}_\varepsilon \subset X_\Omega^\varepsilon \cap B_C(0) \tag{3.18}$$

such that

$$\begin{aligned} &P(D)C_\Delta\big(\big(X \times J(\delta)\big) \cup \big(Y \times J(\beta)\big)\big) \\ &\quad \supset C_\Delta\big(\big(\tilde{X}_\varepsilon \times J(\delta)\big) \cup \big(\tilde{Y} \times J(\gamma)\big)\big)\big|_{((X \times J(\delta)) \cup (Y \times J(\beta))}. \end{aligned} \tag{3.19}$$

Indeed, for $\varepsilon > 0$ we may choose $\eta > 0$ by Lemma 3.1(b) and then use (3.18) and (3.19) for $(X + \overline{B_\eta(0)})$ and $\eta > 0$ instead of X and $\varepsilon > 0$.

(ii) When solving (3.19) and proving (a) we will use an increasing sequence $0 < \gamma_j$. We may assume that $\gamma_0 \leqslant \gamma < \delta$ is sufficiently small. We will use Lemma 2.1 with

$$Q := \big(X \times J(\delta)\big) \cup \big(Y \times J(\beta)\big) \quad \text{and} \quad L := \big(\tilde{X}_\varepsilon \times J(\delta)\big) \cup \big(\tilde{Y} \times J(\gamma_0)\big).$$

Thus let B be a bounded set in $C_\Delta(Q)'_b$. We have to show that

$$\tilde{B} := \big\{\mu \in C_\Delta(Q)' \mid P(-D)\mu \in B\big\} \text{ is bounded in } C_\Delta(L)'_b. \tag{3.20}$$

(iii) In the first step, Proposition 3.2 is used for $K := X$. Let $\tilde{X} := \tilde{K}$ and $\delta_0 > 0$ be chosen for K as in Proposition 3.2 and choose $C > 0$ such that $\tilde{X} \subset \overline{B_C(0)}$. Set $Z := Y \cup \tilde{X}$ and choose $\tilde{\omega}_1$ as in (3.4) to decompose $\tilde{B}$ for $\tilde{Y} := \tilde{Z}$ from Proposition 3.2 and $0 < \gamma_0 < \delta_0$ as

$$\tilde{B} \subset \tilde{B}_I + \tilde{B}_{II},$$

where

$$\tilde{B}_I \subset C_\Delta\big(\tilde{Y} \times J(\gamma_0)\big)'_b \text{ is bounded,} \tag{3.21}$$

and

$$\begin{aligned} &\tilde{B}_{II} \subset C_\Delta\big(\tilde{X} \times J(\delta)\big)'_b \quad \text{and} \quad P(-D)\tilde{B}_{II} \subset C_\Delta\big(\big(\tilde{X} \times J(\gamma_0)\big) \cup \big(X \times J(\delta)\big)\big)'_b \\ &\quad \text{are bounded,} \end{aligned} \tag{3.22}$$

if $0 < \delta < \delta_0/4$ and $0 < \beta$ is sufficiently small (notice that we may substitute Y by Z in the definition of Q and $\tilde{B}$). Since $\tilde{B}_I$ already satisfies the appropriate bounds for (3.20) by (3.21), we now concentrate on $\tilde{B}_{II}$. Notice that

$$\begin{aligned} \tilde{B}_{II} &\subset \tilde{B} - \tilde{B}_I \subset C_\Delta\big(\big(Q \cup \big(\tilde{Y} \times J(\gamma_0)\big)\big) \cap \big(\tilde{X} \times J(\delta)\big)\big)' \\ &= C_\Delta\big(\big(X \times J(\delta)\big) \cup \big(\tilde{X} \times J(\gamma_0)\big)\big)' \end{aligned} \tag{3.23}$$

since $X \subset \tilde{X} \cup Y = Z \subset \tilde{Z} = \tilde{Y}$ and $\beta < \gamma_0 < \delta$.

(iv) Using (3.22) and (3.23), $\tilde{B}_{II}$ is now decomposed in several steps, always using Proposition 3.2 with the assumption (3.3). Fix $\varepsilon > 0$. Then

$$\{u_\mu \mid \mu \in \tilde{B}_{II}\} \text{ is locally uniformly bounded on } \mathbb{R}^{n+1} \setminus \big(\tilde{X} \times J(\delta)\big)$$

by (3.22) and (2.3). To show (3.20) (and (3.18)), we therefore have to prove uniform bounds for u_μ, $\mu \in \tilde{B}_{II}$, near

$$(x, y) \in \big(\tilde{X} \setminus \big(X_\Omega^\varepsilon\big)^\circ\big) \times \pm[\gamma_0, \delta] \subset \big(\tilde{X} \setminus X_\Omega^{\varepsilon/2}\big) \times \pm[\gamma_0, \delta]. \tag{3.24}$$

Recall that $\tilde{X} \subset Z \subset \tilde{Z} = \tilde{Y}$ and that

$$X_\Omega^{\varepsilon/2} \subset \big(X_\Omega^\varepsilon\big)^\circ \quad \text{since} \quad X_\Omega^{\varepsilon/2} + B_{\varepsilon/2}(0) \subset X_\Omega^\varepsilon.$$

Thus, fix $x \in \tilde{X} \setminus X_\Omega^{\varepsilon/2}$. By the definition of $X_\Omega^{\varepsilon/2}$ there is $\xi \in S_0(X, \Omega_{\varepsilon/2})$ such that $x + \xi \notin \Omega$. Since $\xi \in S_0(X, \Omega_{\varepsilon/2})$, there is a path $\lambda : [0,1] \to S_0(X, \Omega_{\varepsilon/2})$ with $\lambda(0) = 0$ and $\lambda(1) = \xi$ such that $\lambda(t) + X \subset \Omega_{\varepsilon/2}$ for any $t \in [0,1]$. Choose $\varepsilon_0 > 0$ such that

$$\overline{B_{\varepsilon_0}\big(\lambda(t)\big)} + X \subset \Omega_{\varepsilon/2} \quad \text{for any } t \in [0,1].$$

For $K := \lambda([0,1]) + \overline{B_{\varepsilon_0}(0)} + X$ we now choose $\tilde{K}$ and $0 < \delta_1 = \delta_1(\varepsilon, x) \leqslant \delta_0$ as in Proposition 3.2 and then fix $0 < \varepsilon_1 < \varepsilon_0$ such that

$$\tilde{X} \cup \tilde{K} \subset \Omega_{\varepsilon_1}.$$

Let $\delta \leqslant \delta_1/4$. Since $x + \lambda(0) = x \in \tilde{X} \subset \Omega_{\varepsilon_1}$ and $x + \lambda(1) = x + \xi \notin \Omega$, there is $0 \leqslant t_\infty \leqslant 1$ such that

$$B_{\varepsilon_1/8}(x) + \lambda(t_\infty) \subset \Omega \setminus \Omega_{\varepsilon_1}. \tag{3.25}$$

Choose $1 > \eta > 0$ such that

$$\big|\lambda(t) - \lambda(\tau)\big| \leqslant \varepsilon_1/4 \quad \text{if } |t - \tau| \leqslant \eta \text{ and } t, \tau \in [0,1]. \tag{3.26}$$

Define

$$t_1 := \sup\big\{t \in [0,1] \mid \tilde{X} + \lambda(t) \subset \Omega_{\varepsilon_1/4}\big\}.$$

By (3.22) and (3.23) we may now apply Proposition 3.2 to $\lambda(t_1) + X \subset K$ (instead of X), $Z := \lambda(t_1) + \tilde{X}$, $\tilde{Z} := \lambda(t_1) + \tilde{X} + B_{\eta\varepsilon_1/8}(0)$, γ_0 instead of β (!) and

$$\tilde{B}_1 := \tau_{\lambda(t_1)}(\tilde{B}_{II}),$$

where τ_ζ is the shift by ζ. We thus get for $0 < \gamma_0 < \gamma_1 \leqslant \gamma$,

$$\tilde{B}_1 \subset \tilde{B}_{1,I} + \tilde{B}_{1,II}$$

with bounded sets

$$\begin{aligned} &\tilde{B}_{1,I} \subset C_\Delta\big(\lambda(t_1) + \big(\big(\tilde{X} + B_{\eta\varepsilon_1/8}(0)\big) \times J(\gamma_1)\big)\big)_b', \\ &\tilde{B}_{1,II} \subset C_\Delta\big(\big(\big[\lambda(t_1) + \big(\tilde{X} + B_{\eta\varepsilon_1/8}(0)\big)\big] \cap \overline{\Omega_{\varepsilon_1}}\,\big) \times J(\delta)\big)_b' \end{aligned} \tag{3.27}$$

and

$$P(-D)\tilde{B}_{1,II} \subset C_\Delta\Big(\Big(\Big(\big[\lambda(t_1) + \big(\tilde{X} + B_{\eta\varepsilon_1/8}(0)\big)\big] \cap \overline{\Omega_{\varepsilon_1}}\Big) \times J(\gamma_1)\Big) \\ \cup \Big(\big(\lambda(t_1) + X\big) \times J(\delta)\Big)\Big)'_b \tag{3.28}$$

if $0 < \delta \leqslant \delta_1/4$ and γ_0 is sufficiently small. Thus, $\tau^{-1}_{\lambda(t_1)}(\tilde{B}_{1,I})$ satisfies the bounds needed for (3.20) (if $\tilde{Y} \supset \tilde{X} + B_{\eta\varepsilon_1/8}(0)$).

We now continue with $\tau^{-1}_{\lambda(t_1)}(\tilde{B}_{1,II})$ instead of $\tilde{B}_{II}$ and $\tilde{X}_1 := [\lambda(t_1) + \tilde{X} + B_{\eta\varepsilon_1/8}(0)] \cap \overline{\Omega_{\varepsilon_1}} - \lambda(t_1)$ instead of $\tilde{X}$, i.e., we define

$$t_2 := \sup\big\{t \in [0,1] \mid \tilde{X}_1 + \lambda(t) \subset \Omega_{\varepsilon_1/4}\big\}$$

and decompose again with $0 < \gamma_1 < \gamma_2 \leqslant \gamma$ if γ_1 is sufficiently small. Since

$$t_{j+1} \geqslant t_j + \eta$$

in each step by (3.26), we end up with $t_{j_0} = t_\infty$ and $\gamma_{j_0} = \gamma$ after $j_0 \leqslant 1/\eta$ steps. We thus get finitely many conditions on γ_j (and finally on β). The sets $\tau^{-1}_{\lambda(t_j)}(\tilde{B}_{j,I})$ all satisfy the bounds in (3.20) for $j \leqslant j_0$ (if $\tilde{Y} \supset \tilde{X} + B_{\varepsilon_1/8}(0)$). Since also

$$\{u_\mu \mid \mu \in \tilde{B}_{j_0,II}\} \text{ is locally bounded on } \mathbb{R}^{n+1} \setminus (\overline{\Omega_{\varepsilon_1}} \times \mathbb{R}),$$

(3.25) implies that

$$\big\{\tau^{-1}_{\lambda(t_\infty)}(u_\mu) \mid \mu \in \tilde{B}_{j_0,II}\big\} \text{ is locally bounded near } x \times [\gamma, \delta].$$

Since the first set in (3.24) is compact, the proof of (a) is complete.

(b) By (a), $\tilde{X}_\varepsilon$ can be chosen as a subset of $(X_\Omega + B_\varepsilon(0)) \cap B_C(0)$. Using the decomposition from Proposition 3.2 again, one can cut off the components as in (3.17) and use Lemma 2.2 to show that the remaining term is bounded in $C_\Delta(\tilde{Y} \times J(\gamma_0))$. This shows (b).

(c) Given $\xi \in S_0(X, \Omega)$, we find a path $\lambda : [0,1] \to \Omega$ with $\lambda(0) = 0$ and $\lambda(1) = -\xi$ such that

$$\lambda(t) + X \subset \Omega \quad \text{for any } t \in [0,1].$$

Similarly as in the proof of (a) we now shift the set $\tilde{B}$ defined by a bounded set $B \subset C_\Delta((\xi + X) \times J(\delta) \cup (Y \times J(\beta)))'_b$ along λ cutting it in each step (notice that $(\xi + X)$ is shifted to X in this way). Then fixing the constant C for X as in (a), we may proceed shifting and cutting as in (a). Thus C is chosen independent of ξ.

(d) By (a) there is a compact $\tilde{Y} \subset \Omega$ such that $\tilde{B}$ satisfies (3.20). By Lemma 2.2 and the assumptions, $\tilde{B}$ is also bounded in $C_\Delta(Y \times J(\delta))'_b$, hence $\tilde{B}$ is bounded in $C_\Delta((\tilde{X}_\varepsilon \times J(\delta)) \cup (Y \times J(\gamma_0)))'_b$ by (2.4). This proves (d). $\square$

From Theorems 3.3 and 2.3 we immediately obtain the following characterization.

Theorem 3.4. *$P(D)$ is surjective on $A(\Omega)$ if and only if the following condition (A_Ω) is satisfied: for any $\omega \Subset \Omega$ there is $C > 0$ such that for any $\varepsilon > 0$ there is $\delta_0 > 0$ such that for any $\hat{\omega} \Subset \Omega$ with*

$$\hat{\omega} \supset \tilde{\omega}_\varepsilon := \big((\bar{\omega})_\Omega + B_\varepsilon(0)\big) \cap B_C(0),$$

any $\xi \in \hat{\omega} \setminus \tilde{\omega}_\varepsilon$ *and any* $0 < T \leqslant 1$ *there are* $V \in U(\hat{\omega} \times \{T\})$ *and* $E \in C_\Delta(W)$, $W := V \cup (\omega \times]T - \delta_0, T + \delta_0[)$, *such that*

$$P(D)E = G(\cdot - \xi,)|_W.$$

Proof. Condition (A_Ω) is sufficient by Theorem 2.3, since it obviously implies $(\underline{A_\Omega})$ (recall that $(\bar{\omega})_\Omega \subset \Omega_\gamma$ for some $\gamma > 0$ by Lemma 3.1(a), hence $\tilde{\omega}_\varepsilon \Subset \Omega$ for small $\varepsilon > 0$).

To show that (A_Ω) is also necessary, fix $\omega \Subset \Omega$ and set $X := \bar{\omega}$. Choose $C > 0$ for X and then $\delta_0 > 0$ and $\tilde{X}_\varepsilon$ for X and ε by Theorem 3.3(a). Fix $\hat{\omega} \Subset \Omega$ as above and define $Y := \bar{\hat{\omega}}$. Let $\gamma := \min\{T/2, \delta_0/2\}$ and choose $\tilde{Y}$ and $\beta > 0$ by Theorem 3.3(a). For $\xi \in \hat{\omega} \setminus \tilde{\omega}_\varepsilon$ we then have

$$G(\cdot - \xi,) \in C_\Delta\big(\big(\tilde{X}_\varepsilon \times [T - \delta_0, T + \delta_0]\big) \cup \big(\tilde{Y} \times [T - \gamma, T + \gamma]\big)\big),$$

since $T - \gamma \geqslant T/2 > 0$ and since $\xi \notin \tilde{\omega}_\varepsilon \supset \tilde{X}_\varepsilon$ by Theorem 3.3(a). By Theorem 3.3, the equation $P(D)E = G(\cdot - \xi,)$ can be solved near

$$\begin{aligned}&\big(X \times [T - \delta_0, T + \delta_0]\big) \cup \big(Y \times [T - \beta, T + \beta]\big)\\ &\quad \supset W := \big(\omega \times]T - \delta_0, T + \delta_0[\big) \cup \big(\hat{\omega} \times]T - \beta, T + \beta[\big).\end{aligned}$$

The theorem is proved. □

We will now give some direct applications of Theorem 3.4, which are similar to the well-known inheritance of surjectivity for partial differential operators on spaces of C^∞-functions (see, e.g., Section 10.6 in Hörmander [9]).

Corollary 3.5. *If* $P(D)$ *is surjective on* $A(\Omega)$ *then* $P(D)$ *is surjective on* $A(\Omega_\varepsilon)$ *for any* $\varepsilon > 0$.

Proof. We have to check the condition $(\underline{A_{\Omega_\varepsilon}})$ by Theorem 2.3. Thus, let $\omega \Subset \Omega_\varepsilon$. Then $\omega \Subset \Omega_{\tilde{\varepsilon}}$ for some $\tilde{\varepsilon} > \varepsilon$, and we may choose $C > 0$ and define $\tilde{\omega}_{\varepsilon_1}$ for ω and $\varepsilon_1 := (\tilde{\varepsilon} - \varepsilon)/2$ by (A_Ω) and Theorem 3.4. Since $(\bar{\omega})_\Omega \subset \Omega_{\tilde{\varepsilon}}$ by Lemma 3.1(a),

$$\tilde{\omega}_{\varepsilon_1} \subset \big(\Omega_{\tilde{\varepsilon}} + B_{\varepsilon_1}(0)\big) \cap B_C(0) \Subset \Omega_\varepsilon$$

and $(\underline{A_{\Omega_\varepsilon}})$ now follows from (A_Ω). □

For operators on C^∞-functions the following stronger result holds: if $P(D)$ is surjective on $C^\infty(\Omega)$ then $P(D)$ is surjective on $C^\infty(\tilde{\Omega}_\varepsilon)$, where

$$\tilde{\Omega}_\varepsilon := \Omega_\varepsilon \cap B_{1/\varepsilon}(0).$$

This is false for operators on $A(\Omega)$. Indeed, for $\Omega := \mathbb{R}^n$ we have $\tilde{\Omega}_\varepsilon = B_{1/\varepsilon}(0)$, but there are operators $P(D)$, which are surjective on $A(\mathbb{R}^n)$ but not on $A(B_{1/\varepsilon}(0))$ (see Hörmander [8]).

Corollary 3.5 is also interesting from the following point of view: Generally speaking, there are few operators $P(D)$ which are surjective on $A(W)$ for some bounded open set W with C^1-boundary (see Langenbruch [14,15]). Now notice, that Ω_ε may be bounded with C^1-boundary for some eventually large ε, though Ω is not. So Corollary 3.5 and the results from Langenbruch [14,15] may impose additional conditions on surjective operators on $A(\Omega)$. Also, $\partial\Omega_\varepsilon$ may have more outer normal vectors than $\partial\Omega$ thus again imposing extra necessary conditions (see Langenbruch [14,15]).

Next we show that surjectivity is inherited to (the interior of) intersections.

Corollary 3.6. *Let $P(D)$ be surjective on $A(\Omega_j)$ for any $j \in J$. Then $P(D)$ is surjective on $A(\Omega)$ for*

$$\Omega := \left(\bigcap_{j\in J} \Omega_j\right)^\circ.$$

Proof. To show $(\underline{A_\Omega})$, let $\omega \Subset \Omega$. Then $\omega \Subset \Omega_\varepsilon$ for some $\varepsilon > 0$, hence $\omega \Subset (\Omega_j)_\varepsilon$ for any $j \in J$. We first show, that $(\underline{A_\Omega})$ holds for any $\xi \in \Omega \setminus \Omega_{\varepsilon/2}$. Indeed, for $\xi \in \Omega \setminus \Omega_{\varepsilon/2}$ there is $j_0 \in J$ such that $\xi \in \Omega_{j_0} \setminus (\Omega_{j_0})_\varepsilon$. As in the proof of Corollary 3.5 we may now choose $\tilde{\omega}_{j_0} \Subset (\Omega_{j_0})_\varepsilon$ and $\delta_{j_0} > 0$ by $(A_{\Omega_{j_0}})$, Theorem 3.4 and Lemma 3.1(a). Since $\xi \notin \tilde{\omega}_{j_0}$, $(\underline{A_\Omega})$ holds for ξ by $(A_{\Omega_{j_0}})$.

We next apply $(A_{\Omega_{j_1}})$ for some fixed $j_1 \in J$ and obtain $(\underline{A_\Omega})$ for any

$$\xi \in \Omega_{j_1} \setminus \big(B_C(0) \cap (\Omega_{j_1})_\varepsilon\big) \supset \Omega \setminus B_C(0).$$

Thus, $(\underline{A_\Omega})$ is proved. □

By Corollaries 3.5 and 3.6 we immediately see that for any $\varepsilon > 0$ and *any set* $A \subset \mathbb{R}^n$,

$$\begin{aligned} &P(D) \text{ is surjective on } A\big(S(A,\Omega_\varepsilon)^\circ\big) \\ &\quad \text{if } P(D) \text{ is surjective on } A(\Omega). \end{aligned}$$

Indeed, we only have to notice that

$$S(A,\Omega_\varepsilon) = \bigcap_{a\in A} (\Omega_\varepsilon - a).$$

Again, the set $S(A,\Omega_\varepsilon)^\circ$ may be rather small (or even void) if compared with Ω. Finally, surjectivity is inherited to (the interior of) the limit inferior:

Corollary 3.7. *Let $P(D)$ be surjective on $A(\Omega_j)$ for any $j \in J$. Then $P(D)$ is surjective on $A(\Omega)$ for*

$$\begin{aligned} \Omega &:= \liminf_j{}^\circ \Omega_j \\ &:= \big\{\xi \in \mathbb{R}^n \mid \exists \varepsilon > 0\colon\ B_\varepsilon(\xi) \subset \Omega_j \textit{ for any but finitely many } j\big\}. \end{aligned}$$

Proof. If J is finite then

$$\Omega = \liminf_j{}^\circ \Omega_j = \mathbb{R}^n$$

and the claim follows from Theorem 4.4 below since $P(D)$ is surjective on $A(\mathbb{R}^n)$ by that theorem.

Let J be infinite. We again show $(\underline{A_\Omega})$. Fix $\omega \Subset \Omega$. Since $P(D)$ satisfies $(\overline{A_{\mathbb{R}^n}})$ by Theorem 4.4 and Theorem 2.3, there is $C > 0$ such that $(\underline{A_\Omega})$ holds for any $\xi \notin B_C(0)$. Let $\omega \Subset \Omega_\varepsilon$ and $\hat{\omega} \Subset \Omega$. By the above definition of the limit inferior of sets and compactness, there is a finite set $J_0 \subset J$ such that

$$\omega \Subset (\Omega_j)_\varepsilon \quad \text{and} \quad \hat{\omega} \Subset \Omega_j \quad \text{for any } j \in J \setminus J_0.$$

Let $\xi \in \hat{\omega} \setminus \Omega_{\varepsilon/2}$. Then there is an infinite set $I \subset J$ such that $\xi \notin (\Omega_j)_\varepsilon$ for any $j \in I$. Thus, there is $i \in I \setminus J_0$ and the claim follows for ξ by (A_{Ω_i}). □

Corollary 3.7 implies, that $P(D)$ is surjective on $A(\bigcup_{j\in\mathbb{N}} \Omega_j)$ if $P(D)$ is surjective on $A(\Omega_j)$ for any $j \in \mathbb{N}$ and if the sets Ω_j are almost increasing, that is, for any $j \in \mathbb{N}$ and any $x \in \Omega_j$ there are $\varepsilon > 0$ and $k \in \mathbb{N}$ such that

$$B_\varepsilon(x) \subset \Omega_m \quad \text{for any } m \geqslant k.$$

Especially, the converse of Corollary 3.5 also holds, that is, that $P(D)$ is surjective on $A(\Omega)$ if $P(D)$ is surjective on $A(\Omega_\varepsilon)$ for any $\varepsilon > 0$.

We next obtain a result which was first proved by Andreotti and Nacinovich [2] (assuming that Ω is bounded) and Zampieri [25] using Hörmander's method. For this, we recall the notion of tangent half spaces (see Andreotti and Nacinovich [2, p. 157]).

Let Ω be convex and let $x_0 \in \partial\Omega$. For $n \in \mathbb{N}$ set

$$\Omega(x_0, n) := x_0 + n(\Omega - x_0)$$

and define the tangent cone for Ω at x_0 by

$$C(x_0, \Omega) := \bigcup_{n\in\mathbb{N}} \Omega(x_0, n).$$

$C(x_0, \Omega)$ is a convex cone. If it is a half space, it is called a tangent half space, if not, we choose $x_1 \in \partial C(x_0, \Omega)$ with $x_1 \neq x_0$ and define the convex cone $C(x_1, C(x_0, \Omega))$. If $C(x_1, C(x_0, \Omega))$ is not a halfspace, we can take $x_2 \in \partial C(x_1, C(x_0, \Omega))$ with $x_2 \notin \mathbb{R}(x_1 - x_0)$ and define the convex cone $C(x_2, C(x_1, C(x_0, \Omega)))$. Proceeding in this way, we finally end up with a half space. All half spaces constructed in this way are called tangent half spaces for Ω at x_0.

Corollary 3.8. *Let Ω be convex. Then $P(D)$ is surjective on $A(\Omega)$ if and only if $P(D)$ is surjective on $A(H)$ for any tangent half space H of Ω.*

Proof. Since Ω and the tangent half spaces are convex, we can assume that P is homogeneous (see Hörmander [8]).

"$\Rightarrow$" Since $P(D)$ is homogeneous, $P(D)$ is surjective on $A(\Omega(x_0, n))$ for any $x_0 \in \partial\Omega$ and any $n \in \mathbb{N}$. Since the sets $\Omega(x_0, n)$ are increasing with n (see Andreotti and Nacinovich [2, p. 157]), we have

$$\liminf_{n\in\mathbb{N}}{}^\circ \Omega(x_0, n) = \bigcup_{n\in\mathbb{N}} \Omega(x_0, n) = C(x_0, \Omega)$$

and $P(D)$ is surjective on $A(C(x_0, \Omega))$ by Corollary 3.7. Proceeding in this way, $P(D)$ is surjective on $A(H)$ for any tangent half space of Ω.

"$\Leftarrow$" This follows from Corollary 3.6, since Ω is the interior of the intersection of its tangent half spaces by Andreotti and Nacinovich [2]. $\square$

Additionally to the statement in Corollary 3.8, the surjectivity of $P(D)$ on $A(\mathbb{R}^n \setminus \bar{H})$ for all tangent half spaces H was used in Zampieri [25]. This extra condition is redundant. It anyway follows from the surjectivity of $P(D)$ on $A(H)$ since we may assume that $P(D)$ is homogeneous.

For $0 \neq N \in \mathbb{R}^n$ let

$$H_N := \{x \in \mathbb{R}^n \mid \langle x, N\rangle > 0\} \quad \text{and} \quad S^n := \{x \in \mathbb{R}^n \mid \|x\| = 1\}.$$

Corollary 3.9. *The set*

$$\{N \in S^n \mid P(D) \text{ is surjective on } A(H_N)\}$$

is compact and symmetric.

Proof. Let $N_k \in S^n$ be a sequence with limit N and let $P(D)$ be surjective on $A(H_{N_k})$ for any k. Then $P(D)$ is surjective on $A(H_N)$ by Corollary 3.7 since

$$\liminf_k{}^{\circ} H_{N_k} = H_N. \qquad \square$$

In the remaining part of this section we prove some results connected to Corollary 3.8 for non-convex Ω. We first show that surjectivity is inherited to $A(H_N)$ if Ω "almost" contains H_N. To make this precise, we define the cones $\Gamma_k(N)$ for $N \in S^n$ and $k \in \mathbb{N}$ as follows:

$$\Gamma_k(N) := \{x \in \mathbb{R}^n \mid \langle x, N\rangle > \|x\|/k\}.$$

Corollary 3.10. *Let $\Omega \neq \mathbb{R}^n$ and $N \in S^n$ and let $P(D)$ be surjective on $A(\Omega)$. Then $P(D)$ is surjective on $A(H_N)$ if for any $k \in \mathbb{N}$ there is $x_k \in \mathbb{R}^n$ such that $x_k + \Gamma_k(N) \subset \Omega$.*

Proof. (i) For any $k \in \mathbb{N}$ there is $\Omega_k \subset \mathbb{R}^n$ such that $P(D)$ is surjective on $A(\Omega_k)$ and

$$\Gamma_k(N) \subset \Omega_k \quad \text{and} \quad B_{1/k}(0) \setminus \Omega_k \neq \emptyset. \tag{3.29}$$

Choose $x_k \in \mathbb{R}^n$ such that $x_k + \Gamma_k(N) \subset \Omega$ and set

$$t_0 := \inf\{t \in \mathbb{R} \mid tN + x_k + \Gamma_k(N) \subset \Omega\}.$$

Then $t_0 > -\infty$ since $\Omega \neq \mathbb{R}^n$. Moreover,

$$t_0 N + x_k + \Gamma_k(N) \subset \Omega, \quad \text{while } B_{1/k}(0) + t_0 N + x_k + \Gamma_k(N) \nsubseteq \Omega,$$

by the minimality of t_0. We may therefore choose $y_k \in (t_0 N + x_k + \Gamma_k(N))$ such that $B_{1/k}(y_k) \setminus \Omega \neq \emptyset$. The claim then follows for $\Omega_k := \Omega - y_k$ since

$$y_k + \Gamma_k(N) \subset t_0 N + x_k + \Gamma_k(N) \subset \Omega$$

since $\Gamma_k(N)$ is convex.

(ii) Set

$$\Omega_I := \liminf_{k\in\mathbb{N}}{}^{\circ}\,\Omega_k.$$

Then

$$\Omega_I \supset \liminf_{k\in\mathbb{N}}{}^{\circ}\,\Gamma_k(N) = \bigcup_{k\in\mathbb{N}} \Gamma_k(N) = H_N \quad \text{and} \quad 0 \notin \Omega_I \tag{3.30}$$

by (3.29). $P(D)$ is surjective on $A(\Omega_I)$ by Corollary 3.7. Hence, $P(D)$ is surjective on $A(\Omega_{II})$ for

$$\Omega_{II} := \left(\bigcap_{\xi\in N^\perp} \Omega_I - \xi\right)^{\circ}$$

by Corollary 3.6. By (3.30) we have

$$\mathbb{R}^n \setminus N^\perp \supset \Omega_{II} \supset H_N.$$

Thus, H_N is a component of Ω_{II}, and $P(D)$ is also surjective on $A(H_N)$. □

The assumption of Corollary 3.10 can be localized if P_m is homogeneous. This leads to the following definition of inner almost normals:

$$IAN(\partial\Omega) := \bigl\{N \in S^n \mid \forall k \in \mathbb{N},\ \exists \varepsilon > 0,\ x_k \in \partial\Omega\colon\ B_\varepsilon(x_k) \cap \bigl(x_k + \Gamma_k(N)\bigr) \subset \Omega\bigr\}.$$

Notice that we do not assume that Ω has C^1-boundary near some boundary point nor that N is the inner unit normal to $\partial\Omega$ at some boundary point in the usual sense. Of course the latter vectors are contained in $IAN(\partial\Omega)$.

Corollary 3.11. *Let $P(D)$ be homogeneous and surjective on $A(\Omega)$. Then $P(D)$ is surjective on $A(H_N)$ for any $N \in IAN(\partial\Omega)$.*

Proof. For $N \in IAN(\partial\Omega)$ and $k \in \mathbb{N}$ choose x_k and $\varepsilon > 0$ as above. Set $\Omega_k := \Omega - x_k$. Then $0 \in \partial\Omega_k$. Since P is homogeneous, $P(D)$ is surjective on $A(j\Omega_k)$ for any $j \in \mathbb{N}$. By Corollary 3.7, $P(D)$ is surjective on $A(\tilde{\Omega}_k)$ for

$$\tilde{\Omega}_k := \liminf_{j\in\mathbb{N}}{}^{\circ}(j\Omega_k) \supset \liminf_{j\in\mathbb{N}}{}^{\circ}\bigl(j\bigl(B_\varepsilon(0) \cap \Gamma_k(N)\bigr)\bigr) = \Gamma_k(N).$$

Since $0 \notin \tilde{\Omega}_k$ for any $k \in \mathbb{N}$, $\tilde{\Omega}_k$ satisfies (3.29) for any $k \in \mathbb{N}$, and the claim now follows as in the proof of Corollary 3.10. □

Corollary 3.12. *Let $P(D)$ be homogeneous. Then $P(D)$ is surjective on $A(\mathrm{conv}(\Omega))$ if $P(D)$ is surjective on $A(\Omega)$ and if Ω has C^1-boundary.*

Proof. $\mathrm{conv}(\Omega)$ is the interior of the intersection of the half spaces $x_N + H_N$, which contain Ω (here N is the inner unit normal to $\partial\Omega$ at x_N). Since $P(D)$ is surjective on $A(H_N)$ by Corollary 3.11, the claim follows from Corollary 3.6. □

4. Surjectivity of $P(D)$ on $A(\mathbb{R}^n)$

In this section we will show that $P(D)$ is surjective on $A(\mathbb{R}^n)$ if $P(D)$ is surjective on $A(\Omega)$ for some non-void $\Omega \subset \mathbb{R}^n$. For convex Ω this was shown by Hörmander [8].

The proof is based on the results from the previous section, especially on Theorem 3.3, and on the following theorem of Hörmander (see Hörmander [8, Theorem 1.1] and the discussion following Theorem 1.3 in Hörmander [8]).

Theorem 4.1. *Let $W \subset \mathbb{R}^n$ be convex and open and let $z \in W$. $P(D)$ is surjective on $A(\mathbb{R}^n)$ if the following holds: for any $f \in A(W)$ and for any $w \Subset W$ there is $u \in C(w)$ such that $P(D)u = f$ on w, and such that u can be analytically extended to a neighborhood U_z of z in $\mathbb{C}^n$ which is independent of w.*

Also, a careful study of the Ω-hull of points in Ω is needed (recall Section 2 for the respective definitions). We will always assume in this section that

$$0 \in \Omega.$$

Lemma 4.2.

(a) $\{z\}_\Omega = \{x \in \mathbb{R}^n \mid x - z + \Omega \subset \Omega\} = z + \{0\}_\Omega \subset \Omega$ *if* $z \in \Omega$.
(b) $\{0\}_\Omega + \Omega_\varepsilon \subset \Omega_\varepsilon$ *for any* $\varepsilon > 0$.
(c) $\{0\}_\Omega$ *is a closed subsemigroup of* $(\mathbb{R}^n, +)$.
(d) $\{0\}_\Omega = \{x \in \Omega \mid d(x+\xi, \partial\Omega) \geqslant d(\xi, \partial\Omega)$ *for any* $\xi \in \Omega\}$.

Proof. (a) By definition, $S(\{z\}, \Omega) = \Omega - z = S_0(\{z\}, \Omega)$ since Ω is connected. This implies (a), since $z \in \Omega$.

(b) Let $\xi \in \Omega_\varepsilon$. Then $B_{\tilde\varepsilon}(\xi) \subset \Omega$ for some $\tilde\varepsilon > \varepsilon$ and hence

$$\left(\{0\}_\Omega + \xi\right) + B_{\tilde\varepsilon}(0) \subset \{0\}_\Omega + B_{\tilde\varepsilon}(\xi) \subset \{0\}_\Omega + \Omega \subset \Omega$$

by (a) (for $z := 0$), that is,

$$\{0\}_\Omega + \xi \subset \Omega_\varepsilon.$$

(c) Let $x, z \in \{0\}_\Omega$. Then

$$x + z + \Omega \subset x + \Omega \subset \Omega,$$

hence, $x + z \in \{0\}_\Omega$ by (a). Let $x_n \in \{0\}_\Omega$ and $\lim_{n\to\infty} x_n =: x$. Then

$$x + \Omega_\varepsilon \subset \overline{\Omega_\varepsilon} \subset \Omega$$

for any $\varepsilon > 0$ by (b). Hence $x + \Omega \subset \Omega$ and $x \in \{0\}_\Omega$ by (a).

(d) "$\subseteq$" Let $x \in \{0\}_\Omega$ and $\xi \in \Omega$ with $d(\xi, \partial\Omega) = \varepsilon$. Then $x \in \Omega$ by (a), and (b) implies that

$$\xi + x \in \Omega_{\tilde\varepsilon} \quad \text{for any } 0 < \tilde\varepsilon < \varepsilon$$

and therefore

$$d(\xi + x, \partial\Omega) \geqslant \varepsilon = d(\xi, \partial\Omega).$$

"$\supseteq$" We fix x from the right-hand side of (d). For $\xi \in \Omega$ choose a path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = 0$ and $\gamma(1) = \xi$. Then

$$d\big(x + \gamma(t), \partial\Omega\big) \geqslant d\big(\gamma(t), \partial\Omega\big) > 0 \quad \text{for any } t \in [0,1].$$

This implies that $x + \xi = x + \gamma(1) \in \Omega$ since $x + \gamma(0) = x \in \Omega$. Thus, $x \in \{0\}_\Omega$ by (a). □

To apply Theorem 4.1, we need to know, if for some point $K := \{z\}$ in Ω the set $\tilde{K}_\varepsilon$ may be chosen by Theorem 3.3 within a *convex* subset of Ω. This is shown in the last part of the following

Proposition 4.3. *Let $\Gamma(0,\Omega) := \{x \in \{0\}_\Omega \mid tx \in \{0\}_\Omega$ if $t > 0\}$.*

(a) *$\Gamma(0,\Omega)$ is a closed subsemigroup of $\{0\}_\Omega$ and a convex cone.*
(b) *0 is an isolated point of $\{0\}_\Omega$ if $\Gamma(0,\Omega) = \{0\}$.*
(c) *For $0 \neq z \in \mathbb{R}^n$ let π_z denote the orthogonal projection onto $z^\perp$ and let $\Omega' := \pi_z(\Omega)$. Then*

$$\pi_z\big(\{0\}_\Omega\big) \subset \{0\}_{\Omega'}.$$

(d) *For any connected open set $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$ and any $C > 0$ there are $z \in \Omega$ and $\varepsilon > 0$ such that the convex hull of the component of z in*

$$\big(\{z\}_\Omega + B_\varepsilon(0)\big) \cap B_C(z)$$

is a relatively compact subset of Ω.

Proof. (a) The first claim follows from Lemma 4.2(c). Since $\Gamma(0,\Omega)$ is a cone by definition, convexity follows from the semigroup property.

(b) Let $0 \neq x_n \in \{0\}_\Omega$ be a sequence tending to 0. We may assume that $x := \lim_{n\to\infty} x_n/\|x_n\|$ exists. We will show that $x \in \Gamma(0,\Omega) = \{0\}$, a contradiction. Fix $t > 0$ and choose $k(n) \in \mathbb{N}$ such that $t/\|x_n\| \in [k(n), k(n)+1]$. For $\varepsilon > 0$ and $\xi \in \Omega_\varepsilon$ we then get

$$tx + \xi = \lim_{n\to\infty}\big(tx_n/\|x_n\| + \xi\big) = \lim_{n\to\infty}\Big(\big(tx_n/\|x_n\| - k(n)x_n\big) + k(n)x_n + \xi\Big) \in \overline{\Omega_\varepsilon}$$

by Lemma 4.2(b) since $k(n)x_n \in \{0\}_\Omega$ by Lemma 4.2(c) and

$$\big\|k(n)x_n - tx_n/\|x_n\|\big\| \leqslant \|x_n\| \to 0.$$

Hence,

$$tx + \Omega = \bigcup_{\varepsilon>0}(tx + \Omega_\varepsilon) \subset \bigcup_{\varepsilon>0}\overline{\Omega_\varepsilon} = \Omega,$$

that is, $tx \in \{0\}_\Omega$ by Lemma 4.2(a) and $x \in \Gamma(0,\Omega)$.

(c) We may assume that $z = e_n$ is the nth canonical unit vector. Let $x = (x', x_n) \in \{0\}_\Omega$ and $\xi' \in \Omega'$. Then there is $\xi_n \in \mathbb{R}$ such that $(\xi', \xi_n) \in \Omega$. Therefore,

$$(x', x_n) + (\xi', \xi_n) \in \Omega$$

by Lemma 4.2(a) and $(x' + \xi') \in \Omega'$, that is, $x' \in \{0\}_{\Omega'}$ again by Lemma 4.2(a) (notice that also Ω' is open and connected and that $0 \in \Omega'$).

(d) The claim will be proved by induction on n.

For $n = 1$, the claim is trivially satisfied for $z := 0$. Indeed, Ω then is an interval, hence convex, and by Lemma 4.2(a) and (c), $\{0\}_\Omega$ is closed in $\mathbb{R}$ and contained in Ω.

Let the claim be proved for fixed $n \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^{n+1}$ be open with $0 \in \Omega$. We distinguish two cases:

If $\Gamma(0, \Omega) = \{0\}$, then 0 is isolated in $\{0\}_\Omega$ by (b) and the component of $z := 0$ in

$$\big(\{0\}_\Omega + B_\varepsilon(0)\big) \cap B_C(0)$$

is $B_\varepsilon(0)$ for small $\varepsilon > 0$, which shows the claim.

On the other hand, assume that there is $y \in \Gamma(0, \Omega)$ with $\|y\| = 1$.

Let W denote the component of 0 in

$$\big(\{0\}_\Omega + B_\varepsilon(0)\big) \cap B_C(0).$$

$\pi_y(W)$ is connected since π_y is continuous. Since $\|\pi_y\| \leqslant 1$, $\pi_y(W)$ is contained in the component W' of 0 in

$$\big(\{0\}_{\Omega'} + B'_\varepsilon(0)\big) \cap B'_C(0)$$

by (c). Since Ω' is connected and contains 0, Lemma 4.2(a) implies that the component of $y' \in \Omega'$ in $(\{y'\}_{\Omega'} + B'_\varepsilon(0)) \cap B'_C(y')$ is $y' + W'$. We may thus use the inductive hypothesis to get $y' \in \Omega' \subset y^\perp$ such that

$$y' + \overline{\operatorname{conv}(W')} \Subset \Omega'.$$

Since $y \in \Gamma(0, \Omega)$, we may use Lemma 4.2(d) and a compactness argument to show that there are $C_1 > 0$ and $\tilde{\varepsilon} > 0$ such that

$$ty + y' + \overline{\operatorname{conv}(W')} \subset \Omega_{\tilde{\varepsilon}}$$

for any $t \geqslant C_1$. Therefore,

$$y' + (C_1 + C)y + \operatorname{conv}(W) \subset [C_1, C_1 + 2C]y + \big(y' + \operatorname{conv}(W')\big) \Subset \Omega.$$

This proves the claim for $z := y' + (C_1 + C)y$ since $\{z\}_\Omega = z + \{0\}_\Omega$ by Lemma 4.2(a). $\square$

The following theorem is the main result of this paper.

Theorem 4.4. *$P(D)$ is surjective on $A(\mathbb{R}^n)$ if $P(D)$ is surjective on $A(\Omega)$ for some $\Omega \neq \emptyset$.*

Proof. We will apply Theorem 3.3 to check the assumptions of Theorem 4.1. Choose $C > 0$ uniformly for $x \in S_0(\{0\}, \Omega) = \Omega$ by Theorem 3.3(c). Then choose $z \in \Omega$ and $\varepsilon > 0$ by Proposition 4.3(d) such that the convex hull $\tilde{X}$ of the component of z in $(\{z\}_\Omega + B_\varepsilon(0)) \cap B_C(z)$ is a relatively compact subset of Ω. Fix $\eta > 0$ and $\delta_0 > 0$ by Theorem 3.3(a) for $X := \{z\}$. Choose a convex open set W such that $\tilde{X} \Subset W \Subset \Omega$. Let $f \in A(W)$. By means of the Cauchy problem for the Laplacian we may extend f to $f_1 \in C_\Delta(\tilde{X} \times J(\delta))$ for some $0 < \delta < \delta_0$. Let w be open and convex with $\tilde{X} \subset w \Subset W$ and set $Y := \overline{w}$. Choose a convex compact $\tilde{Y} \subset W$ such that $Y \subset \tilde{Y}^\circ$. Again by the Cauchy problem, we know that $f_1 \in C_\Delta(\tilde{Y} \times J(\gamma))$ for some $0 < \gamma < \delta$.

By Theorem 3.3(a), (b) and (d), there is

$$g_1 \in C_\Delta\big(\big(B_\eta(z) \times]-\delta, \delta[\big) \cup \big(w \times]-\beta, \beta[\big)\big)$$

such that $P(D)g_1 = f_1$. Thus,

$$g := g_1|_{w\times\{0\}} \in A(w), \quad P(D)g = f \quad \text{on } w$$

and g can be holomorphically extended to a neighborhood U_z of z since g_1 is harmonic on $B_\eta(z) \times]-\delta, \delta[$. U_z is independent of w since η and δ are independent of w. Thus, the assumptions of Theorem 4.1 are satisfied and $P(D)$ is surjective on $A(\mathbb{R}^n)$ by that theorem. □

Corollary 4.5. *The following are equivalent*:

(a) *There is a non-void open set $\Omega \subset \mathbb{R}^n$ such that $P(D)$ is surjective on $A(\Omega)$.*
(b) *For any open set $\Omega \subset \mathbb{R}^n$ there is a smallest open set $\tilde{\Omega} \subset \mathbb{R}^n$ containing Ω such that $P(D)$ is surjective on $A(\tilde{\Omega})$.*
(c) *$P(D)$ is surjective on $A(\mathbb{R}^n)$.*

Proof. (a) implies (c) by Theorem 4.4. (b) follows from (c) by Corollary 3.6. (b) clearly implies (a). □

We finally notice, that surjectivity of $P(D)$ on $A(\Omega)$ may be split into a global property (namely, surjectivity of $P(D)$ on $A(\mathbb{R}^n)$) and a property of type (A_Ω) for points near the boundary:

Corollary 4.6. *The following are equivalent*:

(a) *$P(D)$ is surjective on $A(\Omega)$.*
(b) *$P(D)$ is surjective on $A(\mathbb{R}^n)$ and for any $\omega \Subset \Omega$ and any $C > 0$ there is $\varepsilon > 0$ such that for any $\xi \in \Omega \cap B_C(0)$ with $d(\xi, \partial\Omega) \leqslant \varepsilon$ and any $\hat{\omega} \Subset \Omega$ there is $\delta > 0$ such that for any $0 < T \leqslant \delta$ there are $V \in U(\hat{\omega} \times \{T\})$ and $E \in C_\Delta(W)$, $W := V \cup (\omega \times]T - \delta, T + \delta[)$, such that*

$$P(D)E = G(\cdot - \xi,)|_W.$$

Proof. "(a) ⇒ (b)" The first statement follows by Theorem 4.4, the second one from $(\overline{A_\Omega})$ and Theorem 2.3.

"(b) ⇒ (a)" We have to check $(\underline{A_\Omega})$ for

$$\omega := \tilde{\Omega}_\gamma := \big\{x \in \Omega \mid d(x, \partial\Omega) > \gamma \text{ and } \|\xi\| < 1/\gamma\big\} \quad \text{for } \gamma > 0.$$

By Theorem 2.3 (for $\Omega = \mathbb{R}^n$), $P(D)$ satisfies $(\overline{A_{\mathbb{R}^n}})$ and hence there is $C > 0$ such that the conditions in $(\underline{A_\Omega})$ are satisfied for ω as above and any $\xi \in \mathbb{R}^n$ with $\|\xi\| \geqslant C$. Choose $\varepsilon > 0$ for C by (b). For $\xi \in \Omega$ with $\|\xi\| < C$ the conditions in $(\underline{A_\Omega})$ then are satisfied by assumption if $d(\xi, \partial\Omega) \leqslant \varepsilon$, that is, if $\xi \notin \Omega_\varepsilon$. This proves $(\underline{A_\Omega})$ for any $\xi \in \Omega \setminus \tilde{\omega}$ for $\tilde{\omega} := B_C(0) \cap \Omega_\varepsilon$. □

We finish the paper by giving some examples and applications of the results of this paper. For this we need some further notation.

An important class of operators for our problem is the class of locally hyperbolic operators defined as follows: P_m is locally hyperbolic if for any $0 \neq \theta \in \mathbb{R}^n$ with $P_m(\theta) = 0$ there are $0 \neq N \in \mathbb{R}^n$ and $C > 0$ such

$$P_m(\theta + x + zN) \neq 0 \quad \text{if } |x| + |z| < C \text{ and } \operatorname{Im}(z) \neq 0.$$

The localization $P_{m,\theta}$ of P_m at a real root $\theta \neq 0$ of P_m is defined by

$$P_{m,\theta}(x) := \lim_{s \to 0}\bigl(P_m(\theta + sx)s^{-q_\theta}\bigr),$$

where q_θ is the lowest order of the expansion (with respect to s) of $P_m(\theta + sx)$.

Finally, let

$$\Lambda(P_m) := \bigl\{x \in \mathbb{R}^n \mid P_m(y + sx) \equiv P_m(y) \text{ for any } y \in \mathbb{R}^n\bigr\}$$

(see Hörmander [9, (10.2.8)]) which obviously is a linear space. If $\Lambda(P_m) \neq \{0\}$ then P_m "depends on fewer variables."

If $P(D)$ is not surjective on $A(\mathbb{R}^n)$ then $P(D)$ is surjective on $A(\Omega)$ for no open set Ω by Theorem 4.4. We notice some examples from the literature.

Example 4.7. $P(D)$ is surjective on $A(\Omega)$ for no open set $\Omega \subset \mathbb{R}^n$ in each of the following cases:

(a) $n = 3$ and P_m is not locally hyperbolic.
(b) $n \geqslant 3$ and P_m is a quadratic form which is neither elliptic nor proportional to a real indefinite quadratic form or the square of a real linear form.
(c) There is $0 \neq \theta \in \mathbb{R}^n$ with $P_m(\theta) = 0$ such that $P_{m,\theta}(D)$ does not have a continuous linear right inverse in $C^\infty(\mathbb{R}^n)$.
(d) $\Lambda(P_m) \neq \{0\}$ and $P_m(D)$ does not have a continuous linear right inverse in $C^\infty(\mathbb{R}^n)$.

Proof. This follows from Theorem 4.4, since $P(D)$ is not surjective on $A(\mathbb{R}^n)$ in each case (see Theorems 6.5 and 6.6 in Hörmander [8] for (a) and (b); (d) is a special case of (c), since $P_{m,\theta} = P_m$ for any $\theta \in \Lambda(P_m)$. (c) follows from Theorem 3.14 in Meise et al. [19] and Satz 6.4.1 and 5.5.2 in Braun [3]). □

The conditions in Example 4.7(d) are, e.g., satisfied if $\Lambda(P_m) \neq \{0\}$ and P_m is elliptic as a polynomial on $\Lambda(P_m)^\perp$.

Operators admitting a continuous linear right inverse in $C^\infty(\Omega)$ have been studied by Meise et al. in a series of papers (see, e.g., Meise et al. [18] for hints to the corresponding literature).

If the principal part of P is the power of a real linear form, a complete characterization of the surjectivity of $P(D)$ can be obtained as follows.

Example 4.8. Let $\Omega \subset \mathbb{R}^n$ be open and let $P_m(D) = \langle N, D\rangle^m$ for some $0 \neq N \in \mathbb{R}^n$. The following are equivalent:

(a) $P(D)$ is surjective on $A(\Omega)$.
(b) Ω is $P(D)$-convex for supports.
(c) $\operatorname{dist}(x, \partial\Omega)$ satisfies the minimum principle on every line $F := \xi + \mathbb{R}N$, that is, for every compact interval $K \subset F \cap \Omega$ we have

$$\min_{x \in K} \operatorname{dist}(x, \partial\Omega) = \min_{x \in \partial_F K} \operatorname{dist}(x, \partial\Omega).$$

(d) $P_m(D)$ is surjective on $A(\Omega)$.

Proof. We may assume that $N = e_1$, the first canonical unit vector.

"(a) $\Rightarrow$ (b)" This holds for general $P(D)$ by Zampieri [24].

"(b) $\Rightarrow$ (c)" Assume that there is a compact interval $K := [a, b]e_1 + \xi \subset F \cap \Omega$ such that

$$\varepsilon := \min_{x \in K} \operatorname{dist}(x, \partial\Omega) < \delta := \min\bigl(\operatorname{dist}(ae_1 + \xi, \partial\Omega), \operatorname{dist}(be_1 + \xi, \partial\Omega)\bigr).$$

By Theorem 12.7.5 of Hörmander [9] there is a C^∞-zerosolution f defined on $\mathbb{R}^n$ such that $\operatorname{supp}(f) = F + B_{\gamma/4}(0)$, $\gamma := \delta - \varepsilon$. Using a suitable cut off function ϕ we obtain

$$\operatorname{dist}\bigl(\operatorname{supp}\bigl(P(D)(f\phi)\bigr), \partial\Omega\bigr) \geqslant \delta - \gamma/4 > \varepsilon + \gamma/4 \geqslant \operatorname{dist}\bigl(\operatorname{supp}(f\phi), \partial\Omega\bigr).$$

Hence, Ω is not $P(D)$-convex for supports.

"(c) $\Rightarrow$ (a)" By the remark after Theorem 12.7.5 of Hörmander [9] $P(D)$ has hyperfunction elementary solutions E_- and E_+ such that $\operatorname{supp}(E_\pm) \subset \pm[0, \infty[$. Let $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) = \varepsilon > 0$. Then $\operatorname{dist}(x, \partial\Omega) \leqslant \varepsilon$ on one of the components of x in $x + (\pm[0, \infty[e_1 \cap \Omega)$ by the minimum principle. We may thus define a hyperfunction $E \in \mathfrak{B}(\Omega)$ by defining E to be the shift of E_+, say, near that component in Ω and setting $E = 0$ on the complement in Ω of the component. This proves (A_Ω) for $\omega := \Omega_\varepsilon$. Since $P(D)$ is surjective on $A(\mathbb{R}^n)$, (a) follows from Corollary 4.6.

"(c) $\Leftrightarrow$ (d)" This is well known since $P_m(D) = \partial_1^m$ (see, e.g., Hörmander [9, Theorem 10.8.5]). $\square$

For general operators and general open sets it is not known if $P(D)$ is surjective on $A(\Omega)$ if and only if $P_m(D)$ is surjective on $A(\Omega)$. For convex Ω this is a result of Hörmander [8].

The heat equation and the Laplacian in two variables (considered as an operator in three variables) were the first examples of operators which are not surjective on $A(\mathbb{R}^3)$ (see Piccinini [21]). We can now completely characterize the surjectivity for this kind of operators.

Example 4.9. Let $n \geqslant 3$ and let $P_m(D) = \Delta_k := \sum_{j=1}^k \partial_j^2$, $1 \leqslant k \leqslant n$, be the Laplacian in k variables.

(a) Let $k = n$. Then $P(D)$ is surjective on $A(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$.
(b) Let $1 < k < n$. Then $P(D)$ is surjective on $A(\Omega)$ for no open set $\Omega \subset \mathbb{R}^n$.
(c) Let $k = 1$. Then $P(D)$ is surjective on $A(\Omega)$ if and only if $\operatorname{dist}(x, \partial\Omega)$ satisfies the minimum principle on every line $F := \xi + \mathbb{R}e_1$.

Proof. (a) This follows from the classical solvability theory in $C^\infty(\Omega)$ since $P(D)$ is elliptic.

(b) This follows from Example 4.7(b).

(c) This follows from Example 4.8(c). $\square$

Notice that for $\Omega \subset \mathbb{R}^2$, $P(D)$ is surjective on $A(\Omega)$ if and only if Ω is $P(D)$-convex for supports (see Zampieri [23] and Langenbruch [16, Corollary 2.11] for an alternative proof).

References

[1] K.G. Andersson, Global solvability of partial differential equations in the space of real analytic functions, in: Coll. on Analysis, Rio de Janeiro, 1972, Analyse fonctionelle, Hermann, Paris, 1974.

[2] A. Andreotti, M. Nacinovich, Analytic Convexity and the Principle of Phragmén–Lindelöf, Scuola Norm. Supérieure, Pisa, 1980.

[3] R.W. Braun, Surjektivität partieller Differentialoperatoren auf Roumieu–Klassen, Habilitationsschrift, Düsseldorf, 1993.

[4] R.W. Braun, The surjectivity of a constant coefficient homogeneous differential operator on the real analytic functions and the geometry of its symbol, Ann. Inst. Fourier 45 (1995) 223–249.

[5] R.W. Braun, R. Meise, B.A. Taylor, The geometry of analytic varieties satisfying the local Phragmen–Lindelöf condition and a geometric characterization of the partial differential operators that are surjective on $A(\mathbb{R}^4)$, Trans. Amer. Math. Soc. 356 (2004) 1315–1383.

[6] R.W. Braun, R. Meise, B.A. Taylor, Surjectivity of constant coefficient partial differential operators on $A(\mathbb{R}^4)$ and Whitney's C_4-cone, Bull. Soc. Roy. Liege 70 (2001) 195–206.

[7] A. Grothendieck, Sur les espaces des solutions d'une classe générale d'équations aux derivées partielles, J. Anal. Math. 2 (1952/53) 243–280.

[8] L. Hörmander, On the existence of real analytic solutions of partial differential equations with constant coefficients, Invent. Math. 21 (1973) 151–183.

[9] L. Hörmander, The Analysis of Linear Partial Differential Operators I–II, in: Grundlehren Math. Wiss., vols. 256–257, Springer, Berlin, 1983.

[10] A. Kaneko, On the global existence of real analytic solutions of linear partial differential equations on unbounded domain, J. Fac. Sci. Tokyo 32 (1985) 319–372.

[11] A. Kaneko, A sharp sufficient geometric condition for the existence of global real analytic solutions on a bounded domain, J. Math. Soc. Japan 39 (1987) 163–170.

[12] A. Kaneko, On the global solvability of linear partial differential equations with constant coefficients in the space of real analytic functions, Banach Center Publ. 33 (1996) 149–160.

[13] T. Kawai, On the global existence of real analytic solutions of linear differential equations I, J. Math. Soc. Japan 24 (1972) 481–517.

[14] M. Langenbruch, Localizations of partial differential operators and surjectivity on spaces of real analytic functions, Studia Math. 140 (2000) 15–40.

[15] M. Langenbruch, Surjective partial differential operators on real analytic functions defined on convex open sets, Manuscripta Math. 103 (2000) 241–263.

[16] M. Langenbruch, Characterization of surjective partial differential operators on spaces of real analytic functions, Studia Math. 162 (2004) 53–96.

[17] F. Mantovani, S. Spagnolo, Funcionali analitici reali e funzioni armoniche, Ann. Scuola Norm. Sup. Pisa (3) 18 (1964) 475–513.

[18] R. Meise, B.A. Taylor, D. Vogt, Continuous linear right inverses for partial differential operators with constant coefficients and Phragmen–Lindelöf conditions, in: K.D. Bierstedt, A. Pietsch, W.M. Ruess, D. Vogt (Eds.), Functional Analysis, in: Lecture Notes in Pure and Applied Math., vol. 150, Dekker, 1994, pp. 357–389.

[19] R. Meise, B.A. Taylor, D. Vogt, Phragmen–Lindelöf principles on algebraic varieties, J. Amer Math. Soc. 11 (1998) 1–39.

[20] T. Miwa, On the global existence of real analytic solutions of systems of linear differential equations with constant coefficients, Proc. Japan Acad. 49 (1973) 500–502.

[21] L.C. Piccinini, Non-surjectivity of the Cauchy–Riemann operator on the space of real analytic functions on $\mathbb{R}^3$, Boll. Un. Mat. Ital. 4 (1973) 12–28.

[22] H.G. Tillmann, Dualität in der Potentialtheorie, Portugal. Math. 13 (1954) 55–86.

[23] G. Zampieri, A sufficient condition for the existence of real analytic solutions of PDE with constant coefficients, in open sets of $\mathbb{R}^2$, Rend. Sem. Mat. Univ. Padova 63 (1980) 83–87.

[24] G. Zampieri, A link between C^∞ and analytic solvability of PDE with constant coefficients, in open sets of $\mathbb{R}^2$, Rend. Sem. Mat. Univ. Padova 63 (1980) 145–150.

[25] G. Zampieri, Propagation of singularity and existence of real analytic solutions of locally hyperbolic equations, J. Fac. Sci. Univ. Tokyo Ser. IA Math. 31 (1984) 373–390.